Kuwait University
Faculty of Science
Department of Mathematics

Math 101: Calculus I
Final Exam, 7 January 2012
Duration 2 hours

Answer all of the following questions. **Show all your work.**
Calculators and mobile phones are not allowed.

1. [2+2 pts.] Evaluate the following limits, if they exist.

   (a) $\lim_{x\to 2} \frac{x^2+1}{x-2}$

   (b) $\lim_{x\to 2} (x-2)^2 \cos\left(\frac{2}{x-2}\right)$

2. [4 pts.] Find all values of the constants a and b for which f is continuous at $x = -1$.

$$f(x) = \begin{cases} \frac{4b}{x-1} & \text{if } x < -1, \\ a+b & \text{if } x = -1, \\ ax^2 + x & \text{if } x > -1. \end{cases}$$

3. [4 pts.] Show that the equation $x^3 - 3x^2 + 5x = 7$ has exactly one real root.

4. [4 pts.] Find the point on the line $x - y = 40$ for which $P = x^2 + y^2$ is minimum.

5. [4 pts.] Find an equation of the tangent line to the graph of $f(x) = \int_2^{3x-x^2} \frac{1}{t^2+4}\,dt$ at $x = 1$.

6. [4 pts.] Find the average value of the function $f(x) = |x-3|$ on the interval $[1, 4]$.

7. [4 pts.] Find the area of the region enclosed by the curves $y = x^2$ and $y = 2x + 3$.

8. [3+3 pts.] Set up an integral for the volume of the solid obtained by rotating the region enclosed by the curves $y = x^2$ and $y = 1$ about each of the lines:

   (a) $y = -2$,

   (b) $x = 2$.

9. [3+3 pts.] Evaluate the following integrals.

   (a) $\int \frac{\sin x}{(3+\cos x)^2}\,dx$

   (b) $\int_{-1}^{1} \left(t^3 \cos t + 2\sqrt{1-t^2}\right) dt$

1. (a) The limit $\boxed{\text{does not exist}}$ since
$$\lim_{x\to 2^-} \frac{x^2+1}{x-2} = -\infty \quad \text{and} \quad \lim_{x\to 2^+} \frac{x^2+1}{x-2} = \infty.$$
(b) For all $x \in \mathbb{R}$ we have $-(x-2)^2 \leqslant (x-2)^2 \cos\left(\frac{2}{x-2}\right) \leqslant (x-2)^2$.
Since $\lim_{x\to 2} -(x-2)^2 = \lim_{x\to 2}(x-2)^2 = 0$, by the squeeze theorem we have
$$\boxed{\lim_{x\to 2}(x-2)^2 \cos\left(\frac{2}{x-2}\right) = 0}$$

2. For f to be continuous at -1, we must have $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(-1)$.
Since $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{4b}{x-1} = -2b$, $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} ax^2 + x = a - 1$, and $f(-1) = a + b$, for continuity at -1 we must have $-2b = a + b$ and $a + b = a - 1$. The second equation gives $\boxed{b = -1}$, and substituting this in the first equation we get $\boxed{a = 3}$.

3. Let $f(x) = x^3 - 3x^2 + 5x - 7$. Then f is continuous and differentiable on $\mathbb{R}$ since it is a polynomial function.
We have $f(0) = -7 < 0$ and $f(3) = 8 > 0$. So by the intermediate value theorem, the equation $f(x) = 0$ has at least one real root.
Now suppose that the equation $f(x) = 0$ has at least two roots a and b with $a < b$. Then $f(a) = f(b) = 0$ and by Rolle's theorem, there exists $c \in (a, b)$ such that $f'(c) = 0$. That is $3c^2 - 6c + 5 = 0$. This is a contradiction since $\Delta = 6^2 - 4 \times 3 \times 5 = -24 < 0$ implies the equation $3c^2 - 6c + 5 = 0$ has no real root.

4. Substituting $y = x - 40$ in P we obtain $P = P(x) = x^2 + (x-40)^2$. We then have $P'(x) = 2x + 2(x - 40)$. Since $P'(x)$ is always defined, critical numbers of $P(x)$ are roots of $P'(x)$. Therefore, 20 is the only critical number of $P(x)$. On the other hand, $P''(x) = 4 > 0$, so $P(x)$ has a local minimum at 20. Since P has only one critical number, $P(x)$ indeed has its absolute minimum at $x = 20$. So P is minimized at the point $\boxed{(20, -20)}$.
An alternate way for proving that P is minimized at $x = 20$ is to notice that $P'(x) < 0$ for all $x < 20$ and $P'(x) > 0$ for all $x > 20$. Hence P is decreasing on $(-\infty, 20)$ and increasing on $(20, \infty)$, which implies P is minimized at $x = 20$, that is at the point $(20, -20)$.

5. By the fundamental theorem of calculus and the chain rule we obtain $f'(x) = \dfrac{3-2x}{(3x-x^2)^2+4}$.
Evaluating at $x = 1$ we get $f'(1) = 1/8$ and $f(1) = \int_2^2 1/(t^2+4)\,dt = 0$. Hence an equation for the tangent line is $\boxed{y - 0 = \frac{1}{8}(x-1)}$.

6. We have

$$f(x) = \begin{cases} -(x-3) & \text{if } x < 3, \\ x-3 & \text{if } x \geqslant 3. \end{cases}$$

Now

$$f_{\text{ave}} = \frac{1}{4-1}\int_1^4 f(x)\,dx = \frac{1}{3}\left[\int_1^3 (3-x)\,dx + \int_3^4 (x-3)\,dx\right] = \frac{1}{3}\left(2+\frac{1}{2}\right) = \boxed{\frac{5}{6}}.$$

7. For points of intersection of the two curves we solve $x^2 = 2x+3$. This gives $x^2-2x-3 = (x+1)(x-3) = 0$, which holds for $x=-1$ and $x=3$. Evaluating the two functions at an intermediate value, say $x=0$, we see that $y=2x+3$ is the top boundary and $y=x^2$ is the bottom boundary of the region. The area of the region is then calculated as

$$A = \int_{-1}^{3} \left((2x+3)-x^2\right)dx = \left(x^2+3x-\frac{x^3}{3}\right)_{-1}^{3} = \boxed{\frac{32}{3}}.$$

8. We first find the points of intersection of the two curves $y=x^2$ and $y=1$. Solving $x^2=1$ we obtain $x=\pm 1$. This gives the points $(-1,1)$ and $(1,1)$. For integration with respect to y, we must notice that the minimum value of $y=x^2$ in the region of interest is at the point $(0,0)$.

   (a) Integrating with respect to x, we must use the method of washers and we obtain

   $$V_1 = \int_{-1}^{1} \pi\Big[(1+2)^2 - (x^2+2)^2\Big]\,dx.$$

   On the other hand, integrating with respect to y we must use the method of cylindrical shells to obtain

   $$V_1 = \int_0^1 2\pi(y+2)(2\sqrt{y})\,dy.$$

   (b) Integrating with respect to x, we must use the method of cylindrical shells and we obtain

   $$V_2 = \int_{-1}^{1} 2\pi(2-x)(1-x^2)\,dx.$$

   Integrating with respect to y we must use the method of washers to obtain

   $$V_2 = \int_0^1 \pi\Big[(2+\sqrt{y})^2 - (2-\sqrt{y})^2\Big]\,dy.$$

9. (a) Let $u = 3+\cos x$. Then $du = -\sin x\,dx$ and we have

   $$\int \frac{\sin x}{(3+\cos x)^2}\,dx = -\int \frac{1}{u^2}\,du = \frac{1}{u}+C = \boxed{\frac{1}{3+\cos x}+C}.$$

   (b) $\int_{-1}^{1} t^3\cos t\,dt = 0$ since $t^3\cos t$ is an odd function of t, and $\int_{-1}^{1}\sqrt{1-t^2}\,dt = \frac{\pi}{2}$ since it represents the area of a semicircle with radius 1 centered at the origin. Putting these together, we obtain

   $$\int_{-1}^{1}\left(t^3\cos t + 2\sqrt{1-t^2}\right)dt = 0 + 2\cdot\frac{\pi}{2} = \boxed{\pi}$$